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SHORTER COMMUNICATIONS

TWO NEW THEOREMS IN HEAT CONDUCTION

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NOMENCLATURE

A, area of heat transfer [ft^2];
C_p, specific heat;
D, limit of integration denoting whole system;
E_T, Thévenin equivalent e.m.f. [V];
f, g, any real functions;
G, any scalar;
I, loss operator;
J, operator form of Υ;
K, thermal conductivity [Btu/h ft°F];
K_1, K_2, K_3, thermal conductivities of three different materials in sample application [Btu/h ft°F];
L, H, region dimensions in example calculation [ft];
ℓ, distance between two node points [ft];
Q, limit of integration denoting source region;
q''', volumetric source strength [Btu/h ft^3];
R, limit of integration denoting perturbed region;
R_T, Thévenin equivalent resistance [Ω];
$\mathscr{R}$, resistance to heat flow [°F h)/Btu];
$\mathbf{r}$, vector denoting position;
r_1, radius of source wire in sample application [ft];
r_2, distance from source wire axis to surface of strengthening wire in sample application [ft];
r_3, distance from source wire axis to outermost surface of strengthening wire in sample application [ft];
S, surface limit of integration denoting surface of whole system;
T, operator form of t;
$\bar{T}$, average source temperature [°F];
t, temperature [°F];
$\mathbf{V}$, any vector;
x, space variable in example calculation [ft].

Greek symbols

α, β, arbitrary constants;
δ, variation operator;
$\delta\bar{\lambda}$, perturbation in $\bar{\lambda}$ when $\bar{\lambda}$ is taken to be an eigenvalue;
$\delta\Lambda$, limit of $\delta\lambda$;
θ, argument denoting time;
$\bar{\theta}$, particular time;
$\bar{\lambda}$, ratio of losses to sources for whole system;
ξ, limit of integration denoting some part of system;
σ, surface of ξ;
Υ, complimentary function;
φ, arbitrary linear operator;
∇, gradient operator;
γ, characteristic angle of strengthening wire in sample application.

Superscripts

p, perturbed state;
*, adjoint function;
0, unperturbed.

Subscripts

p, perturbed state;
0, steady state.

INTRODUCTION

FREQUENTLY, in parametric survey and design feasibility studies, one is interested in some parameter that characterizes the overall performance rather than the detailed behaviour of the system in question. Examples of such parameters would be the effective multiplication constant for nuclear reactors or the total heat-transfer rate for heat exchangers. This paper describes the development and application of theorems which may be used to relate changes in the parameters that characterize an exclusively steady-state heat conducting system (geometry, conductivities and source strengths) to the average source temperature in all source regions. If this source temperature is taken as the characteristic index of the system performance, several areas of practical application are possible.

The theorems as presented here are mathematically general and may be applied to any system that can be described by the heat-conduction equation. Their derivation was motivated by a desire to explore the physical significance of the eigenvalue of a Sturm–Liouville equation having a solution identical to the heat-conduction equation. When both equations have identical solutions, it is possible to obtain additional information about a system, because the Sturm–Liouville equation involves both an eigenvalue and eigenfunction (instead of just a function). This additional information is obtained by performing classical operations on the eigenvalue equation and physically interpreting the results.

1. FORMULATION OF THE EQUIVALENT STURM–LIOUVILLE PROBLEM

Consider a bounded heat conduction system D composed of fixed heat generating (source) regions and heat dissipating regions. If linearity of the conduction equation is assumed, the temperature at each point in this system may be described by

$$\nabla.K(\mathbf{r})\,\nabla t(\mathbf{r},\theta) + \dot{q}'(\mathbf{r}) = \rho(\mathbf{r})\,C_p(\mathbf{r})\frac{\partial t(\mathbf{r},\theta)}{\partial\theta}. \tag{1.1}$$

The temperature distribution defined by (1.1) at any particular time $\hat{\theta}$ may also be envisioned as a steady-state eigenfunction solution of the suitably formed three dimensional (space only) Sturm–Liouville equation

$$-\nabla.K(\mathbf{r})\,\nabla t(\mathbf{r},\hat{\theta}) = \bar{\lambda}(\hat{\theta})\,\Upsilon(\mathbf{r},\hat{\theta})\,t(\mathbf{r},\hat{\theta}) \tag{1.2}$$

where the complementary function $\Upsilon(\mathbf{r},\hat{\theta})$ is appropriately chosen. The only restriction placed on (1.2) is that at least one eigenfunction must be identical to the temperature distribution obtained from (1.1) for any selected value of $\hat{\theta}$. Integration of (1.2) over the system volume gives

$$-\int_D \nabla.K(\mathbf{r})\,\nabla t(\mathbf{r},\hat{\theta})\,dV = \bar{\lambda}(\hat{\theta})\int_D \Upsilon(\mathbf{r},\hat{\theta})\,t(\mathbf{r},\hat{\theta})\,dV. \tag{1.3}$$

Since the heat flux $K(\mathbf{r})\,\nabla t(\mathbf{r},\hat{\theta})$ is continuous for all $\hat{\theta}$ (except possibly zero), the left side of (1.3) may be transformed by Gauss's theorem to surface integrals of the heat flux over all parts of the system and is recognized as the total heat loss rate from all parts the system at time $\hat{\theta}$. Thus,

$$\begin{aligned}-\sum_{\text{all }\xi}\int_\xi \nabla.K(\mathbf{r})\,\nabla t(\mathbf{r},\theta)\,dV &= -\sum_{\text{all }\sigma}\int_\sigma K(\mathbf{r})\,\nabla t(\mathbf{r},\hat{\theta})\,d\sigma\\ &= \text{system heat loss rate at time }\hat{\theta}\\ &= \bar{\lambda}(\hat{\theta})\int_D \Upsilon(\mathbf{r},\hat{\theta})\,t(\mathbf{r},\hat{\theta})\,dV\end{aligned} \tag{1.4}$$

where ξ denotes some part of the system, and the sum of the ξ equals D. The summation has been used to allow for possible discontinuities in the heat flux at $\hat{\theta} = 0$. The eigenvalue is now defined to be the ratio of the sum of the heat loss rates to heat production rate for the entire system at time $\hat{\theta}$:

$$\lambda(\hat{\theta}) \equiv \frac{\sum_{\text{all }\xi}\int_\xi K(\mathbf{r})\,\nabla t(\mathbf{r},\hat{\theta})\,d\mathbf{S}}{\int_D \dot{q}'(\mathbf{r})\,dV} = \frac{\text{losses/s}}{\text{productions/s}} \tag{1.5}$$

This definition fixes the integral of $\Upsilon(\mathbf{r},\hat{\theta})t(\mathbf{r},\hat{\theta})$ as

$$\int_D \Upsilon(\mathbf{r},\hat{\theta})\,t(\mathbf{r},\hat{\theta})\,dV = \int_D \dot{q}'(\mathbf{r})\,dV = \text{total system heat source} \tag{1.6}$$

The form of $\Upsilon(\mathbf{r},\hat{\theta})$ is not known for all values of $\hat{\theta}$; however, it can be seen that, for the steady state ($\theta = \infty$), $\bar{\lambda}$ is 1·0 and the functional form of $\Upsilon(\mathbf{r},\infty)$ must be

$$\Upsilon(r,\infty) = q(r)/t(r,\infty). \tag{1.7}$$

The steady-state form of this complementary function, given by (1.7), thus transforms the nonhomogeneous steady-state conduction equation into a homogeneous eigenvalue form with known linear operators.

2. PERTURBING THE EIGENVALUE

The eigenvalue equation, (1.2), may be thought of as describing a fictitious system, which shall be called the equivalent eigensystem. The following derivation is concerned with the imbalance in the eigensystem, designated as $\delta\bar{\lambda}$, associated with alterations in the conductivity in any region of a geometrically arbitrary system.

For simplicity, the system is assumed to be at the steady state. For this condition, the $\hat{\theta}$ argument is dropped, and equation (1.1) becomes

$$-\nabla.K(\mathbf{r})\,\nabla t(\mathbf{r}) = \dot{q}'(\mathbf{r}). \tag{2.1}$$

For this case the eigenvalue of the eigentemperature distribution defined by (1.5) and (1.7) is 1.0 and is denoted by $\bar{\lambda}_0$. Equation (1.2) is then

$$-\nabla.K(\mathbf{r})\,\nabla t_0(\mathbf{r}) = \bar{\lambda}_0\,\Upsilon(\mathbf{r})\,t_0(\mathbf{r}). \tag{2.2}$$

To facilitate the manipulations that follow, this is rewritten in operator form as

$$IT_0 = \bar{\lambda}_0 JT_0 \tag{2.3}$$

where,

$$I \equiv -\nabla.K(r)\nabla \tag{2.4}$$

$$J \equiv \Upsilon(\mathbf{r}) \tag{2.5}$$

$$T_0 \equiv t_0(r) \tag{2.6}$$

(zero subscript denotes the unperturbed steady state).

Next consider the adjoint eigentemperature, T_0^*, defined by

$$I^*T_0^* = \bar{\lambda}_0 J^*T_0^* \tag{2.7}$$

which is subject to the same boundary conditions as the solution of (2.3).

If the system experiences some change in physical properties of some subregion R contained in D, the loss of operator

I of region R will change by an amount, $\delta\bar{\lambda}$, and the eigentemperature distribution by δT. In order that the source remain unchanged J must be altered by an amount δJ such that

$$\int_D (J + \delta J)(T_0 + \delta T)\,dV = \int_D \bar{q}'(r)\,dV. \tag{2.8}$$

The defining equation for the perturbed system is then

$$(I + \delta I)(T_0 + \delta T) = (\bar{\lambda}_0 + \delta\bar{\lambda})(J + \delta J)(T_0 + \delta T). \tag{2.9}$$

Multiplication of the perturbed equation (2.9) on the left by the unperturbed adjoint eigentemperature T_0^*, and the unperturbed adjoint equation (2.7) by the perturbed temperature $(T_0 + \delta T)$ gives

$$T_0^*(I + \delta I)(T_0 + \delta T) = (\bar{\lambda}_0 + \delta\bar{\lambda})\,T_0^*(J + \delta J)(T_0 + \delta T) \tag{2.10}$$

and

$$(T_0 + \delta T)\,I^*T_0^* = \bar{\lambda}_0(T_0 + \delta T)\,J^*T_0^*. \tag{2.11}$$

Subtracting (2.11) from (2.10), integrating the result over the system volume, and making use of the identity

$$\int f\varphi g\,dV = \int g\varphi^* f\,dV \tag{2.12}$$

(where f and g are any real functions satisfying the same homogeneous boundary conditions, and φ is a linear operator) give for $\delta\bar{\lambda}$ exactly,

$$\delta\bar{\lambda} = \frac{\int_R T_0^*\delta I(T_0 + \delta T)\,dV - (\bar{\lambda}_0 + \delta\bar{\lambda})\int_Q T_0^*\delta J(T_0 + \delta T)\,dV}{\int_Q T_0^* J(T_0 + \delta T)\,dV} \tag{2.13}$$

where Q denotes the source region. If the limit is taken in all integrals, the second integral in the numerator vanishes by equations (2.8) and (1.6) giving exactly

$$\delta\Lambda \equiv \frac{\int_R T_0^*\delta I T_0\,dV}{\int_Q T_0^* J T_0\,dV} = \lim_{\delta T\to 0}(\delta\lambda). \tag{2.14}$$

The admissibility of the limiting operation is a point that may be arbitrary for the analytic version of (2.14); this was not explored. However, for numerical representations of the integrals in (2.14) this operation is correct, and the primary source of application of the theorems to be derived is numerical.

It should be observed that if the perturbation is made over most of Q (i.e. if $R \simeq Q$)$\delta\Lambda$ may be approximated using relatively crude estimates of the temperature distribution.

The error in the $\delta\Lambda$ so calculated will be much less than the error in the temperatures used for evaluation [1] because when $R \simeq Q$, (2.14) is a stationary form for $\delta\Lambda$. That is, $\delta\Lambda$ in this case may be taken as the eigenvalue of the equation

$$\delta I T_0 = \delta\Lambda J T_0. \tag{2.15}$$

Since the limiting form $\delta\Lambda$ does not involve J or δJ, the zero subscript in all of the preceding derivation may refer to an initial distribution which has not reached the steady state.

3. TEMPERATURE PREDICTION—THEOREMS I AND II

A utilitarian aspect of the eigenvalue approach to heat conduction is the prediction of change in average source temperature resulting from a change in conductivity or source anywhere in the system. In the previous discussion, the operator J was taken to be fixed and associated with a particular temperature distribution and time. For the purposes of the development that follows, this operator is thought of as varying continuously at all times following the perturbation in such a manner as to insure that the real temperature and eigentemperature remain identical.

a. Conductivity effects

The eigenvalue variation is given in operator notation by (2.13), which for the purpose of this derivation will be considered to incorporate a time dependence in T, J, and $\bar{\lambda}$. Following a conductivity perturbation, as θ approaches infinity, $\delta\bar{\lambda}$ must become zero since the system tends to regain the steady-state condition. This means that δJ ultimately takes on some nontrivial form in view of the requirement of a continuously varying J operator. If the final temperature distribution, $T_0 + \delta T$, is denoted by T_p, the perturbed steady-state distribution, (2.13) becomes in the limit as $\delta\lambda$ goes to zero

$$\frac{\int_R T_0^*\delta I T_p\,dV}{\int_Q T_0^* J T_p\,dV} = \frac{\bar{\lambda}_0\int_Q T_0^*\delta J T_p\,dV}{\int_Q T_0^* J T_p\,dV} \tag{3.1}$$

The adjoint and real eigentemperatures are identical, equation (2.3) being self adjoint [2]. (The self adjointness of the homogeneous Sturm–Liouville equation only holds for homogeneous boundary conditions of the form $\alpha\,\partial T_0/\partial r + \beta T_0 = 0$. These conditions are zero temperature, zero heat flux, or zero external sink temperature. The present treatment is for boundary conditions of this type. The treatment of nonhomogeneous boundary conditions will be left for subsequent discussion in Appendix A.) Since the system is initially at steady state, $\bar{\lambda}_0$ is 1·0. Replacing T_p by T_0 in the arbitrary denominator of (3.1), dropping the adjoint distinction and making use of (2.1), (2.2), (2.5) and (2.6) give

$$\frac{\int_R T_0\delta I T_p\,dV}{\int_Q \bar{q}'T_0\,dV} = \frac{\int_Q \bar{q}'(T_0 - T_p)\,dV}{\int_Q \bar{q}'T_0 dV}. \tag{3.2}$$

We may replace T_p by T_0 exactly in the numerator of the left side of equation (3.2) if the intergrand is multiplied by the ratio of the unperturbed to perturbed conductivities in region R. This is only valid, however, when the heat flow to all points in the perturbed region remains constant. If the heat source to the perturbed points changes as a result of

the perturbation, a different correction may be required. This is discussed in Appendix B. Making this replacement gives

$$\frac{\int_R T_0 \delta I (K_0/K_p)\, T_0 \, dV}{\int_Q \bar{q}''' T_0 dV} = \frac{\int_Q \bar{q}'''(T_0 - T_p)\, dV}{\int_Q \bar{q}''' T_0 \, dV}. \tag{3.3}$$

Comparing (3.3) and (2.14) it is noted that if the correction, K_p/K_0, is taken outside of the integral, as would be possible if the conductivities were constants over R, equation (3.3) could be written,

$$\delta A = \frac{\dfrac{K_p}{K_0} \int_Q \bar{q}'''(T_0 - T_p)\, dV}{\int_Q \bar{q}''' T_0 \, dV} \tag{3.4}$$

where δA is given by (2.14). Also, if the source strength is spatially constant, we may divide numerator and denominator of (3.4) by the source volume and rearrange to obtain the fractional change in the average temperature of the source due to a conductivity perturbation anywhere in the system: This gives theorem I:

$$\frac{\delta \bar{T}}{\bar{T}} = \left(\frac{K_0}{K_p}\right) \delta A = \left(\frac{K_0}{K_p}\right) \frac{\int_R \delta K |\nabla T_0|^2 \, dV}{\int_Q \bar{q}''' T_0 \, dV}. \tag{3.5}$$

This theorem relates the fractional temperature change in the source region to the limiting form of the eigenvalue change that is caused by an alteration in the loss operator of the equivalent eigensystem.

The conductivity ratio has been assumed constant over R to simplify the derivation. Although this is not necessary, it would be true for any numerical model. If the source is not spatially constant we obtain a different type of average change—namely, the change in the source weighted temperatures.

An interesting corollary implied by (3.5) is

$$\frac{\delta T_1}{\delta \bar{T}_2} = \frac{(K_p)_2}{(K_p)_1} \frac{\delta K_1}{\delta K_2} \tag{3.6}$$

where the subscripts denote different changes in the same region. This enables comparisons of the relative effects on source temperature when using several different materials in a specific region of the same system with no knowledge of the unperturbed system geometry, materials, or source strength.

b. Source effects

The effect of changes in the source distribution on the unperturbed average source temperature can be obtained by manipulations like those in (3.1–3.4). Since integration is considered over the whole system instead of with the limits shown in (3.1),

$$\delta \bar{\lambda} = \int_D T_0^* \delta J (T_0 + \delta T)\, dV = 0 \tag{3.7}$$

This expression includes both the change in source distribution due to the perturbation and the change (in J) in the unperturbed (original) source that is required to offset the temperature change that occurs as a result of the peturbation. Distinguishing these source regions by their integral limits, (3.7) may be written

$$\int_R T_0^* \delta J (T_0 + \delta T)\, dV = -\int_Q T_0^* \delta J (T_0 + \delta T)\, dV$$
$$= -\int_Q \bar{q}'''(T_p - T_0)\, dV. \tag{3.8}$$

If the sources are spatially constant, manipulation of (3.8) and replacing T_0 and T_0^* with t_0 give theorem II:

$$\frac{\delta \bar{T}}{\bar{T}} = \frac{\delta \bar{q}''' \int_R t_0 \, dV}{\bar{q}''' \int_Q t_0 \, dV}. \tag{3.9}$$

Theorem II relates the fractional change in source temperature (the source here being defined as all regions containing sources before the perturbation) to the addition or deletion of a source in any region R.

The corollary to theorem II is formed as was the corollary to theorem I:

$$\frac{\delta \bar{T}_1}{\delta \bar{T}_2} = \frac{(\delta \bar{q}''')_1}{(\delta \bar{q}''')_2}. \tag{3.10}$$

This allows a direct comparison of the average temperature rise in the source region of a system which would result from the addition of sources of varying strength in a given location, with no knowledge of the system properties or any temperature redistribution that may result from the introduction of the extraneous sources.

4. ELEMENTARY EXAMPLE CALCULATIONS

To demonstrate the temperature prediction techniques proposed here, sample calculations are performed for an elementary geometry. These calculations serve the dual purpose of illustrating the method in detail and providing analytical comparisons necessary to complete the exposition. To examine theorem I, the infinite slab at steady state (see Fig. 1) is considered. After a conductivity perturbation in region 1 a conventional calculation gives,

$$\frac{\delta \bar{T}}{\bar{T}_0} = \frac{\delta K_1}{K_1^0} \left[\frac{1}{1 + (3HK_1^0/LK_2^0)}\right]. \tag{4.1}$$

and theorem I gives

$$\frac{\delta\bar{T}}{\bar{T}_0} = \frac{\left(\frac{K_1^0}{K_1^p}\right)(K_1^0 - K_1^p)\frac{'''^2}{(K_1^0)^2}\int_0^L x^2\,dx}{\int_0^L \frac{'''^2 L^2}{2K_1^0}\,dx - \int_0^L \left(\frac{'''^2}{2K_1^0}\right)x^2\,dx + \frac{'''^2 LH}{K_2^0}\int_0^L dx} = \frac{\delta K_1}{K_1^p}\left[\frac{1}{1 + \frac{3HK_1^0}{LK_2^0}}\right]. \tag{4.2}$$

It is noted that for a conductivity perturbation in the non-source region, both methods give

$$\frac{\delta\bar{T}}{\bar{T}_0} = \frac{\delta K_2}{K_2^p}\left[\frac{1}{1 + (LK_2^0/3HK_1^0)}\right]. \tag{4.3}$$

To minimize notation, theorem II will be examined by assuming that the conductivities are the same in all regions. Let the perturbation consist of adding a source to region 2 having the same unit strength as in region 1 before the perturbation. Theorem II gives for the fractional change in $\bar{T}_1$

$$\frac{\partial\bar{T}}{\bar{T}_0} = \frac{\delta q'''\int_R T_0\,dV}{\int_Q q'''\,T_0\,dV} = \frac{\int_L^{L+H} T_2^0\,dx}{\int_0^L T_1^0\,dx} = \frac{3H^2}{2L^2 + 6LH}. \tag{4.4}$$

A conventional computation of the average temperature change gives

$$\frac{\delta\bar{T}}{\bar{T}_0} = \frac{\bar{T}_p - \bar{T}_0}{\bar{T}_0} = \frac{\frac{q'''}{2K}(L+H)^2 - \frac{L^2}{3} - \frac{q'''}{2K}\frac{2}{3}L^2 + 2LH}{\frac{q}{2K}(\frac{2}{3}L^2 + 2LH)} = \frac{3H^2}{2L^2 + 6LH}. \tag{4.5}$$

5. APPLICATIONS

It is not the intent of this paper to elaborate on the application of the theorems proposed here since the *a priori* practicality of any concept is difficult to assess depending strongly, as it does, on individual ingenuity. It is worthwhile, however, to devote some discussion to possible areas of application for the sake of completeness.

The method described herein is most powerful as a practical tool when applied to condition problems requiring numerical solution. A disadvantage associated with numerical calculations is that as some parameter is varied, several machine calculations must be performed in order to establish a trend. The eigenvalue method requires only one calculation of the temperature distribution for a reference case, and an additional calculation of the volume integrals of the gradients squared (which would require negligible computer time compared to the temperature calculation). With this information several types of parameter variation computations could be performed by hand using theorem I.

Effects of nonlinearities are also amenable to hand calculation using theorem I. Suppose that due to extreme temperature variations the thermal conductivity varies strongly with position. To approximate the average source temperature, the temperature distribution is first calculated assuming temperature-independent conductivities. Then the deviations in the source temperature due to the nonlinearity in K are summed.

Another possible application of theorem I is the calculation of changes in one region that are required to offset changes that are made in another region in order for the source temperature to remain constant.

If the relative effect of several different materials in one specific region is of interest, the corollaries may be applied directly.

The efficiency of theorem I, even for problems which are not necessarily numerical, can be seen by considering the following particular example. Suppose that we have a heat source in the form of electric current flow in a wire. This wire is surrounded by a thick electrical insulator as shown in Fig. 2. The cable (wire and insulator) is to be subjected to an axial stress. It is suspected that this tension may break the wire. Hence, we wish to strengthen the cable by imbedding an additional wire in the insulator. This additional wire will not have any electric current flowing in it. We wish to know how much the average temperature of the electrical wire will change due to the presence of the strengthening wire (which has a different conductivity than the insulator material). It is presumed here that knowledge of the change in the average temperature of the electrical wire is desired so that the correct electrical resistance may be assigned to this wire in calculations of current flow.

A direct solution for the average temperature distribution

in the electric wire would be difficult since the presence of the eccentric wire poses formidable boundary considerations. If, however, the radial temperature distribution is first computed assuming that the strengthening wire is not present, the effect of introducing the strengthening wire may be computed as a perturbation.

If it is assumed that the cable surface is at a constant temperature and that there is no axial conduction, the problem may be represented as a two-dimensional conduction problem (see Fig. 2). The absolute change in the

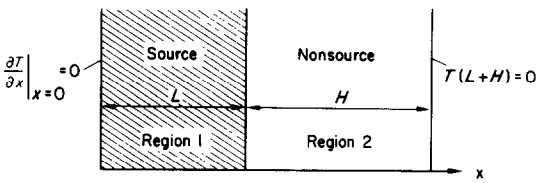

FIG. 1. Composite slab for example calculation.

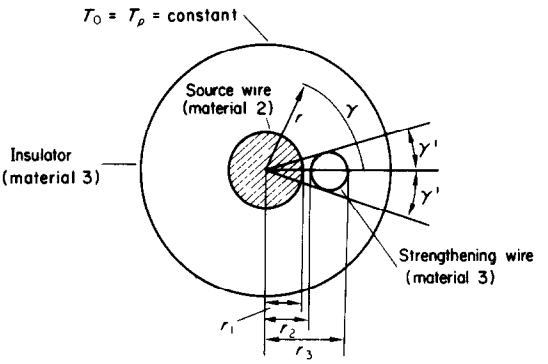

FIG. 2. Geometry of sample application.

average temperature of the source wire due to changing the conductivity in region 3 from that of the insulator material to that of the strengthening wire is

In order to apply these methods to realistic problems, one should be aware of the assumptions that have been made throughout the derivations. The accuracy of the computed result will depend on the extent to which these assumptions are true for the application being considered. These are as follows:

(1) The boundary of the system was assumed to be at the same constant temperature at all points before and after the perturbation. This restriction applies to both theorems I and II.

(2) If the perturbation is large, the heat flow to the perturbed points must not be affected by the perturbation if the method is to be considered exact. For an approximate solution or for small perturbations this restriction is not necessary. This applies only to theorem I.

With these two considerations in mind, some judgement may be exercised in evaluating results computed using these theorems.

6. CONCLUDING REMARKS

It is expected that these theorems may find practical application for conduction systems much like the perturbation formulae of nuclear reactor physics. They are especially suited to geometrically complex systems that must be analyzed numerically in that they enable certain types of parameter variation studies to be performed by hand using the computer solution for a single reference case. These theorems were developed out of a recognition of the physical significance of the eigenvalue of an equivalent Sturm–Liouville solution. Their extended application to other physical systems which can be described by the heat conduction equation requires only a physical interpretation of $\delta\Lambda$.

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APPENDIX A

Evaluation of $\delta\Lambda$ for Conductivity or Source Changes

The denominator of (2.14) presents no problem since by

$$\delta\bar{T} = \bar{T}\left(\frac{\delta\bar{T}}{\bar{T}}\right) \approx \frac{\bar{T}K_3(K_3-K_2)}{K_2\bar{q}'} \frac{\int_{-\gamma'}^{+\gamma'}\int_{r_2}^{r_3} |dT(r,\gamma)/dr|^2\, r\, dr\, d\gamma}{\int_0^{2\pi}\int_0^{r_1} T(r,\gamma)\, r\, dr\, d\gamma} \approx \left(\frac{K_3-K_2}{K_3K_2}\right)\frac{\bar{q}'r_1^2\gamma'\ln(r_3/r_2)}{2\pi}. \tag{5.1}$$

equations (2.1), (2.5), and (2.6) it is just

$$\text{denominator} = \int_Q t_0(\mathbf{r})\, \dddot{q}(\mathbf{r})\, dV. \tag{A.1}$$

The integrand of the numerator is derived as follows:

$$I = -\nabla.K\nabla \tag{A.2}$$

$$I + \delta I = -\nabla.(K + \delta K)\nabla. \tag{A.3}$$

By identity,

$$\nabla.G\mathbf{V} = \nabla G.\mathbf{V} + G\nabla.\mathbf{V} \tag{A.4}$$

where G is any scalar and $\mathbf{V}$ any vector. Using (A.4) yields (A.3) as

$$I + \delta I = -[\nabla(K + \delta K).\nabla + (K + \delta K)\nabla^2]. \tag{A.5}$$

Also, (A.2) can be written

$$I = -[\nabla K.\nabla + K\nabla^2]. \tag{A.6}$$

Subtracting (A.6) from (A.7) gives

$$\delta I = -[\nabla \delta K.\nabla + \delta K\nabla^2]. \tag{A.7}$$

Substituting (A.7) and (A.1) into (2.14), and changing T_0^* and T_0 to their equivalent, t_0, give

$$\delta\Lambda = \frac{-\int_R [t_0\nabla\delta K.\nabla t_0 + t_0\delta K\nabla^2 t_0]\, dV}{\int_Q \dddot{q} t_0\, dV}. \tag{A.8}$$

Making use of the same identity as before,

$$\nabla.(t_0\delta K\nabla t_0) = \nabla t_0\delta K.\nabla t_0 + t_0\delta K\nabla^2 t_0$$
$$= \delta K|\nabla t_0|^2 + t_0\delta K\nabla^2 t_0 + t_0\nabla\delta K.\nabla t_0.$$

Solving for $-(t_0\nabla\delta K.\nabla t_0)$,

$$-(t_0\nabla\delta K.\nabla t_0) = -\nabla.(t_0\delta K\nabla t_0) + \delta K|\nabla t_0|^2 + t_0\delta K\nabla^2 t_0. \tag{A.9}$$

Substituting (A.9) into (A.8) gives

$$\delta\Lambda = \frac{-\int_R [\nabla.(t_0\delta K\nabla t_0)]\, dV + \delta K\int_R |\nabla t_0|^2\, dV}{\int_Q \dddot{q} t_0\, dV}. \tag{A.10}$$

At this point we recall that the boundaries of the system were assumed to be at zero temperature. Then by Gauss's theorem, the first integral in the numerator of (A.10) can be transformed to a surface integral (over the surface, S, of D) which is zero because of the assumed zero surface temperature. Hence, for the zero surface temperature condition,

$$\delta\Lambda = \frac{\int_R \delta K|\nabla t_0|^2\, dV}{\int_Q \dddot{q} t_0\, dV}. \tag{A.11}$$

If the boundary temperatures are nonzero, the application of (A.11) in (3.5) requires that the temperature be constant at the physical boundaries of the system. It can be seen that, for this case, the gradient is unaltered and the denominator is increased by the constant surface temperature. The fractional temperature change in the source $\delta\bar{T}/\bar{T}$ is affected identically by the addition of a constant to the temperatures involved. Hence this is only a normalization effect and (A.11) remains valid for use in (3.5) to predict fractional average source temperature changes.

The $\delta\Lambda$ computed using (A.11) does not represent the imbalance of an eigensystem having nonzero boundaries. If the boundary temperature is some constant, however, t_0 in (A.11) is taken to be the solution to the homogeneous boundary value problem formed by subtracting the boundary value from the temperature [3]. All of the operations required to derive (A.11) may be performed exactly as shown on this temperature excess function (which has been normalized to zero at the boundaries) and (A.11) will again be obtained. Hence, to use (A.11) for the nonhomogeneous boundary condition problem (prescribed nonzero constant) we note that the addition or subtraction of a constant to the numerator results in no change; however, the denominator must be decreased appropriately. The exact expression for the nonzero boundary case is then

$$\delta\Lambda = \frac{\int_R \delta K|\nabla t_0|^2\, dV}{\int_Q \dddot{q} t_0\, dV - t_0(S)\int_Q \dddot{q}\, dV}. \tag{A.12}$$

This expression is never used, however, since this correction is not required for the $\delta\Lambda$ that appears in equation (3.5). Equation (3.9) does, however, require slight modification when used with a constant nonzero boundary problem. For this case the numerator should be corrected to give

$$\frac{\delta\bar{T}}{\bar{T}} = \frac{\int_R \delta\dddot{q} t_0\, dV - t_0(S)\int_R \delta\dddot{q}\, dV}{\int_Q \dddot{q} t_0 dV}. \tag{A.13}$$

APPENDIX B

Replacement of Perturbed Gradient by Unperturbed Gradient in Equation (3.3)

The gradient is numerically a simple function of the temperature differences between any node and its immediate neighbors. A correction to one of these ΔT (that will account for a change in the conductivity of the material separating the two node points) will be examined here primarily to give some insight into the behavior of the gradient correction and to provide qualitative bounds on the range of validity of the temperature prediction method when used as an approximation (without correction).

From elementary direct-current circuit theory, the voltage drop (which is analogous to the temperature difference) across any resistance R is given by

$$\Delta V = \left[\frac{E_T}{R_T + R}\right] R \tag{B.1}$$

where

E_T, Thévenin equivalent e.m.f.;
R_T, Thévenin equivalent resistance of the system measured across the points in question in the absence of R and any source.

The ratio of perturbed to unperturbed voltage drops, and thus temperature differences by analogy, across the same points is then

$$\frac{\Delta T_p}{\Delta T_0} = \frac{\mathscr{R}_p}{\mathscr{R}_0}\,\frac{[\mathscr{R}_T + \mathscr{R}_0]}{[\mathscr{R}_T + \mathscr{R}_p]} = \frac{\left[\frac{\mathscr{R}_T}{\mathscr{R}_0} + 1\right]}{\left[\frac{\mathscr{R}_T}{\mathscr{R}_p} + 1\right]} \tag{B.2}$$

where the $\mathscr{R}$ values refer to thermal resistances

$$\mathscr{R} \equiv \frac{\ell}{AK} \tag{B.3}$$

with ℓ being the separation distance between the points and A the area of heat transfer.

Qualitatively, a perturbation can be characterized by the volume over which is applies and the amount by which the conductivity has been changed. Some insight into the behavior of the correction for limiting combinations of these characteristics can be evaluated using (B.3) if the problem is such that we may assume the following:

(a) Perturbation over a small volume $\Rightarrow \mathscr{R}_T \gg \mathscr{R}_0$ or $\mathscr{R}_p$.
(b) Small perturbation in $K \Rightarrow \mathscr{R}_0 \approx \mathscr{R}_p$.
(c) Perturbation over a large volume $K \Rightarrow \mathscr{R}_T \ll \mathscr{R}_0$ or $\mathscr{R}_p$.
(d) Large change in $\Rightarrow \mathscr{R}_p \ll \mathscr{R}_0$ or the reverse.

Considering all combinations of the above,

Combination	$\Delta T_p/\Delta T_0$
Large volume, small change in K	$\approx 1{\cdot}0$
Large volume, large change in K	$\approx 1{\cdot}0$
Small volume, small change in K	$\approx 1{\cdot}0$
Small volume, large change in K	$\approx \mathscr{R}_p/\mathscr{R}_0 = K_0/K_p$

If the Thévenin equivalent resistances cannot be computed, the above considerations should provide some confidence as to the validity of the result.

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RECALCULATION OF THE SPALDING FUNCTION FOR CONTINUOUS AND SOURCE HEAT FLUX

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NOMENCLATURE

ρ, density;
τ, frictional stress at wall;
c_f, friction factor, $\tau/\frac{1}{2}\rho u_1^2$;
c_p, specific heat at constant pressure;
H, heat flux parameter, $\dot{q}''/\rho u_1 c_p \sqrt{(c_f/2)}$;
Pr, Prandtl number;
$\dot{q}''$, heat flux from wall;
S_q, Spalding function for continuous heat flux;
S_q', Spalding function for source heat flux;
St, Stanton number;
T, temperature difference referred to free stream temperature;
u_1, free stream velocity;
u^+, non-dimensional velocity;
x^+, non-dimensional distance along the wall;
y^+, non-dimensional distance perpendicular to the wall.

1. INTRODUCTION

THE EXACT solution to the problem of predicting the temperature field within the fully developed turbulent boundary layer has been formulated by Spalding with the fundamental assumption of a particular "law of the wall" [1]. Numerical solutions for the cases of constant wall temperature and constant wall heat flux parameter have been obtained by Kestin and Gardner [2] and Smith and Shah [3].